$$Ax^2y'' + Bxy' + Cy = 0$$

GUESS $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$$Ax^2y'' + Bxy' + Cy = \left(Ar(r-1) + Br + C\right)x^r = 0$$

$$Ar(r-1) + Br + C = 0$$

$$Ar^2 - Ar + Br + C = 0$$

$$Ar^2 + (B-A)r + C = 0$$ ← INDICIAL POLYNOMIAL

IF $r = r_1, r_2$

$y = C_1 x^{r_1} + C_2 x^{r_2}$

SANITY CHECK: $x^2y'' + 7xy' - 7y = 0 \rightarrow \boxed{r^2 + 6r - 7 = 0}$

$A = 1,\ B = 7,\ C = -7$

$(r+7)(r-1) = 0$

$\boxed{r = -7, 1}$

$\boxed{y = C_1 x^{-7} + C_2 x}$

BUT WHAT IF

$r_1 = r_2$ OR

$r_1 = a + bi$

AND $r_2 = a - bi$

?

$$Ax^2y'' + Bxy' + Cy = 0 \longrightarrow Ay'' + (B-A)y' + Cy = 0$$

$y = f(x)$

WHERE $y = f(t)$

LET $x = e^t \longrightarrow t = \ln x$

CONSTANT COEF'S HOMOGENEOUS

$$Ar^2 + (B-A)r + C = 0$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) ?$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)\cdot\frac{dy}{dt} + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx} = \frac{d^2y}{dt^2}\cdot\frac{1}{x}$$

$$= -\frac{1}{x^2}\cdot\frac{dy}{dt} + \frac{1}{x}\cdot\frac{1}{x}\frac{d^2y}{dt^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

$$Ax^2y'' + Bxy' + Cy = 0$$

$$= Ax^2\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) + Bx\left(\frac{1}{x}\frac{dy}{dt}\right) + Cy$$

$$= A\frac{d^2y}{dt^2} - A\frac{dy}{dt} + B\frac{dy}{dt} + Cy = A\frac{d^2y}{dt^2} + (B-A)\frac{dy}{dt} + Cy = 0$$

IF THE INDICIAL POLYNOMIAL HAS

① 2 REAL ROOTS (r_1, r_2), $r_1 \neq r_2$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$y = c_1 (e^t)^{r_1} + c_2 (e^t)^{r_2}$

$y = c_1 x^{r_1} + c_2 x^{r_2}$

② 1 DOUBLE REAL ROOT (r_1, r_1)

$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

$y = c_1 (e^t)^{r_1} + c_2 t (e^t)^{r_1}$

$y = c_1 x^r + c_2 x^r \ln x$

③ 2 COMPLEX CONJUGATE ROOTS $(a \pm bi)$

$y = c_1 e^{at} \sin bt + c_2 e^{at} \cos bt$

$y = c_1 x^a \sin(b \ln x) + c_2 x^a \cos(b \ln x)$

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SOLVE $x^2y'' + 5xy' + 4y = 0 \longrightarrow \boxed{r^2 + 4r + 4 = 0}$

$A=1$, $B=5$, $C=4$

$\boxed{r = -2, -2}$

$\boxed{y = C_1 x^{-2} + C_2 x^{-2} \ln x}$

SOLVE $x^2y'' + 7xy' + 13y = 0 \longrightarrow \boxed{r^2 + 6r + 13 = 0}$

$r = \dfrac{-6 \pm \sqrt{36-52}}{2} = \dfrac{-6 \pm 4i}{2}$

$= \boxed{-3 \pm 2i}$

$$\boxed{\begin{aligned} y = {} & C_1 x^{-3} \sin(2 \ln x) \\ & + C_2 x^{-3} \cos(2 \ln x) \end{aligned}}$$

4.4 NON-HOMOGENEOUS LDE

GIVEN $y'' + py' + qy = g$ $\qquad$ p, q, g ARE FUNCTIONS OF y's INDEPENDENT VARIABLE

(LET $L[y] = y'' + py' + qy$)

TH'M: IF y_h IS A SOL'N OF $y'' + py' + qy = 0$

AND y_p IS A PARTICULAR SOL'N OF $y'' + py' + qy = g$

THEN $y_h + y_p$ IS ALSO A SOL'N OF $y'' + py' + qy = g$

PROOF: $L[y_h] = 0$ AND $L[y_p] = g$

SO $L[y_h + y_p] = L[y_h] + L[y_p] = 0 + g = g$

SO $y_h + y_p$ IS A SOL'N OF $y'' + py' + qy = g$

IF y_{p_1} AND y_p ARE BOTH SOL'NS OF $y''+py'+qy=g$

$$
\begin{aligned}
L[y_{p_1}-y_p] &= L[y_{p_1}+(-1)y_p] \\
&= L[y_{p_1}]+(-1)L[y_p] \\
&= g+(-1)g \\
&= 0
\end{aligned}
$$

IE. $y_{p_1}-y_p$ IS A SOL'N OF THE CORRESPONDING HOMOGENEOUS LDE (CALL IT y_h)

$$y_{p_1}-y_p=y_h$$

$$y_{p_1}=y_h+y_p$$

IE. EVERY SOL'N OF THE NON-HOMOGENEOUS LDE CAN BE WRITTEN AS y_p PLUS A SOL'N OF THE CORRESPONDING HOMOGENEOUS LDE

IE. IF GENERAL SOL'N OF $y''+py'+qy=0$ IS $y=C_1y_1+C_2y_2$

THEN $C_1y_1+C_2y_2+y_p$ IS THE GENERAL SOL'N OF THE NON-HOMOGENEOUS LDE

SOLVE $2z''+z=9e^{2x}$

METHOD OF UNDETERMINED COEFFICIENTS

① SOLVE $2z''+z=0$

$$2r^2+1=0 \longrightarrow r^2=-\frac{1}{2}$$

$$r=\pm\frac{\sqrt{2}}{2}i$$

$$z_h = C_1\sin\left(\frac{\sqrt{2}}{2}x\right)+C_2\cos\left(\frac{\sqrt{2}}{2}x\right)$$

② GUESS $z_p = Ae^{2x}$ $(A\in\mathbb{R})$

$$z_p' = 2Ae^{2x}$$

$$z_p'' = 4Ae^{2x}$$

$$2z_p''+z_p = 8Ae^{2x}+Ae^{2x} = 9Ae^{2x} = 9e^{2x}$$

$$A=1$$

$z_p = e^{2x}$ ← YOU: CHECK THIS IS A SOL'N OF $2z''+z=9e^{2x}$

GENERAL SOL'N OF $2z'+z=9e^{2x}$

$$y=y_h+y_p$$

$$y=C_1\sin\left(\frac{\sqrt{2}}{2}x\right)+C_2\cos\left(\frac{\sqrt{2}}{2}x\right)+e^{2x}$$